1次元可解カオス力学系の超離散化とトロピカル幾何

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Schröder map と tent map (1)

sn, cn, dn: **Jacobi の楕円函数** $\quad \mathrm{cn}^2(u;k) = 1 - \mathrm{sn}^2(u;k), \quad \mathrm{dn}^2(u;k) = 1 - k^2\mathrm{sn}^2(u,k), \quad 0 < k < 1$

倍角公式: $$\mathrm{sn}(2u;k) = \frac{2\mathrm{sn}(u;k)\,\mathrm{cn}(u;k)\,\mathrm{dn}(u;k)}{1 - k^2\mathrm{sn}^4(u;k)}$$

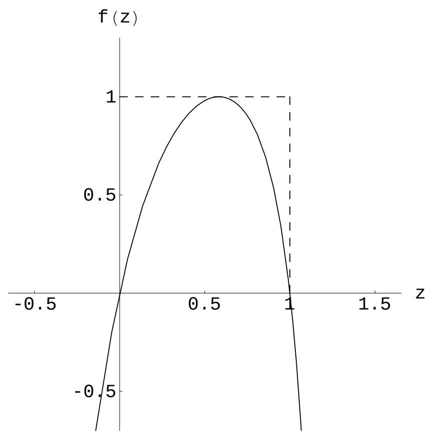
Schröder map

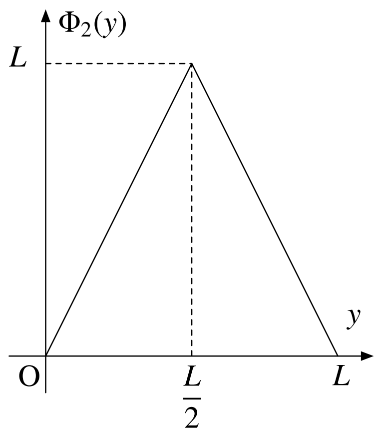
$$z_{n+1} = f(z_n) = \frac{4z_n(1-z_n)(1-k^2z_n)}{(1-k^2z_n^2)^2} \qquad z_n = \mathrm{sn}^2(2^n\xi_0;k), \quad z_n \in [0,1]$$

Tent map

$$y_{n+1} = \Phi_2(y_n) = L\left(1 - 2\left|\frac{y_n}{L} - \frac{1}{2}\right|\right) = \begin{cases} 2y_n & 0 \le y_n \le \frac{L}{2} \\ -2y_n + 2L & \frac{L}{2} \le y_n \le L \end{cases}$$

$k \to 0$: Logistic map $\qquad z_{n+1} = 4z_n(1-z_n), \quad z_n = \sin^2(2^n x_0), \quad z_n \in [0,1]$





Schröder map と tent map (2)

Schröder map と Tent map との共役性:

$$f(z_n) = \frac{4z_n(1-z_n)(1-k^2 z_n)}{(1-k^2 z_n^2)^2}, \qquad \Phi_2(y_n) = L\left(1 - 2\left|\frac{y_n}{L} - \frac{1}{2}\right|\right) = \begin{cases} 2y_n & 0 \le y_n \le \frac{L}{2} \\ -2y_n + 2L & \frac{L}{2} \le y_n \le L \end{cases}$$

$$\phi \circ f \circ \phi^{-1} = \Phi_2, \quad \phi(z) = \frac{L}{K(k)} \mathrm{sn}^{-1}(\sqrt{z}; k), \quad K(k): \text{ 第1種完全楕円積分}$$

$(\because)$

$$0 \le y < \frac{L}{2} \longrightarrow \Phi_2(y) = 2y, \quad z = \phi^{-1}(y) = \mathrm{sn}^2\left(\frac{K}{L}y; k\right)$$

$$(\phi^{-1} \circ \Phi_2)(y) = \mathrm{sn}^2\left(\frac{K}{L}\Phi_2(y); k\right) = \mathrm{sn}^2\left(2\frac{K}{L}y; k\right) = \frac{4z(1-z)(1-k^2 z)}{(1-k^2 z^2)^2} = f(z) = (f \circ \phi^{-1})(y)$$

$$\frac{L}{2} \le y_n \le L \longrightarrow \Phi_2(y) = 2L - 2y, \quad z = \phi^{-1}(y) = \mathrm{sn}^2\left(\frac{K}{L}y; k\right)$$

$$(\phi^{-1} \circ \Phi_2)(y) = \mathrm{sn}^2\left(\frac{K}{L}\Phi_2(y); k\right) = \mathrm{sn}^2\left(2K - \frac{2Ky}{L}; k\right) = \mathrm{sn}^2\left(2\frac{K}{L}y; k\right)$$

$$= f(z) = (f \circ \phi^{-1})(y) \quad (\because \mathrm{sn}^2 \text{の周期 } 2K(k),\ \mathrm{sn}^2: \text{ even})$$

今回の話の概要

✓ **結果の概要**

☛ **Schröder map** と **tent map** の新しい関係

超離散化によって**Schröder map**の写像函数と解がセットで **tent map** に移行する.

☛ **tent map** の新しい幾何学的解釈

あるトロピカル曲線に付随したトロピカルヤコビ多様体上の倍角写像

✓ **話の流れ**

☛ 超離散化: **Burgers** 方程式を例として

Burgers 方程式 ⟶ 離散**Burgers** 方程式

⟶ 超離散 **Burgers** 方程式・**Burgers cellular automaton**・**Rule 184 ECA**

☛ **Schröder map** の超離散化

☛ **Tent map** のトロピカル幾何学的構造

Burgers 方程式

Burgers 方程式: $$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 0$$

☞ 1次元弱非線形散逸系の基本方程式：乱流・交通流・衝撃波⋯

☞ 線形化可能: **Cole-Hopf** 変換 $$u = \frac{1}{f}\frac{\partial f}{\partial x} \longrightarrow \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \longrightarrow f = 1 + \sum_{j=1}^{N} e^{p_j x + p_j^2 t + \xi_{j,0}} \longrightarrow$$ 衝撃波解

離散化：

☞ 線形化されたレベルで離散化 $$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \Longrightarrow \frac{f_n^{t+1} - f_n^t}{\delta} = \frac{f_{n+1}^t - 2f_n^t + f_{n-1}^t}{\epsilon^2}$$

☞ 「離散 **Cole-Hopf**変換」 $$u_n^t = c\,\frac{f_{n+1}^t}{f_n^t}$$

discrete Burgers 方程式: $$u_n^{t+1} = u_{n-1}^t \,\frac{1 + \frac{1-2A}{cA}\,u_n^t + \frac{1}{c^2}\,u_n^t u_{n+1}^t}{1 + \frac{1-2A}{cA}\,u_{n-1}^t + \frac{1}{c^2}\,u_{n-1}^t u_n^t}, \quad A = \frac{\delta}{\epsilon^2}$$

$$f_n^t = 1 + \sum_{j=1}^{N} e^{k_j n + \omega_j t + \xi_{j,0}}, \quad \omega_j = \log(1 + A(e^{k_j} - 2 + e^{-k_j})) \longrightarrow$$ 衝撃波解

Burgers 方程式の超離散化 (1)

Key formula:

$$\lim_{\epsilon \to +0} \epsilon \log\left(e^{\frac{A}{\epsilon}} + e^{\frac{B}{\epsilon}} + \cdots\right) = \max(A, B, \cdots)$$

変数変換: $u_n^t = e^{\frac{U_n^t}{\epsilon}}, \quad c^2 = e^{\frac{L}{\epsilon}}, \quad \dfrac{1-2A}{cA} = e^{-\frac{M}{\epsilon}}$

$$u_n^{t+1} = u_{n-1}^t \, \frac{1 + \frac{1-2A}{cA}\, u_n^t + \frac{1}{c^2}\, u_n^t u_{n+1}^t}{1 + \frac{1-2A}{cA}\, u_{n-1}^t + \frac{1}{c^2}\, u_{n-1}^t u_n^t}$$

$$\Longrightarrow U_n^{t+1} - U_{n-1}^t = \epsilon \log\left(1 + e^{\frac{U_n^t - M}{\epsilon}} + e^{\frac{U_n^t + U_{n+1}^t - L}{\epsilon}}\right) - \epsilon \log\left(1 + e^{\frac{U_{n-1}^t - M}{\epsilon}} + e^{\frac{U_{n-1}^t + U_n^t - L}{\epsilon}}\right)$$

$$\Big\Downarrow \quad \epsilon \to +0$$

$$U_n^{t+1} - U_{n-1}^t = \max\left(0, U_n^t - M, U_n^t + U_{n+1}^t - L\right) - \max\left(0, U_{n-1}^t - M, U_{n-1}^t + U_n^t - L\right)$$

ultradiscrete Burgers 方程式: $U_n^{t+1} = \min\left(M, U_{n-1}^t, K - U_n^t\right) - \min\left(M, U_n^t, L - U_{n+1}^t\right) + U_n^t$

✓ $M, L, U_n^0 \in \mathbb{Z} \quad \Longrightarrow \quad U_n^t \in \mathbb{Z} \qquad \longrightarrow$ 従属変数の離散化！

✓ $\times \longrightarrow +, \quad + \longrightarrow \max$

Burgers 方程式の超離散化 (2)

超離散化のポイント: 線形化可能性や解などの構造とセットで離散化できること

$$u_n^{t+1} = u_{n-1}^t \frac{1 + \frac{1-2A}{cA} u_n^t + \frac{1}{c^2} u_n^t u_{n+1}^t}{1 + \frac{1-2A}{cA} u_{n-1}^t + \frac{1}{c^2} u_{n-1}^t u_n^t} \quad \rightarrow \quad u_n^t = c\frac{f_{n+1}^t}{f_n^t} \quad \rightarrow \quad f_n^{t+1} = Af_{n+1}^t + (1-2A)f_n^t + Af_{n-1}^t$$

$$\Downarrow \quad u_n^t = e^{\frac{U_n^t}{\epsilon}}, \quad c^2 = e^{\frac{L}{\epsilon}}, \quad \frac{1-2A}{cA} = e^{-\frac{M}{\epsilon}}, \quad f_n^t = e^{\frac{F_n^t}{\epsilon}}, \quad \epsilon \rightarrow +0$$

$$\begin{aligned} U_n^{t+1} &= \min\left(M, U_{n-1}^t, L - U_n^t\right) \\ &- \min\left(M, U_n^t, L - U_{n+1}^t\right) + U_n^t \end{aligned} \quad \rightarrow \quad U_n^t = F_{n+1}^t - F_n^t + \frac{L}{2} \quad \rightarrow \quad F_n^{t+1} = \max\left(F_{n-1}^t, F_n^t + \frac{L}{2} - M, F_{n+1}^t\right)$$

解：

$$u_n^t = c\frac{f_{n+1}^t}{f_n^t} = c\frac{1 + e^{k(n+1)+\omega t+\xi_0}}{1 + e^{kn+\omega t+\xi_0}}, \quad \omega_j = \log(1 + A(e^{k_j} - 2 + e^{-k_j}))$$

$$\Downarrow \quad k = \frac{K}{\epsilon}, \omega = \frac{\Omega}{\epsilon}, \xi_0 = \frac{\Xi_0}{\epsilon}, \quad \epsilon \rightarrow +0$$

$$U_n^t = \frac{L}{2} + \max\left(0, K(n+1) + \Omega t + \Xi_0\right) - \max\left(0, Kn + \Omega t + \Xi_0\right)$$

$$\Omega = \lim_{\epsilon\rightarrow+0} \epsilon \log\left(1 + \frac{1}{2 + e^{\frac{L-2M}{2\epsilon}}}\left(e^{\frac{K}{\epsilon}} - 2 + e^{-\frac{K}{\epsilon}}\right)\right) = \max(K, -K) = |K|$$ (assuming $L - 2M < 0$)

Burgers 方程式の超離散化 (3): セルオートマトン化

ultradiscrete Burgers 方程式: $$U_n^{t+1} = \min\left(M, U_{n-1}^t, L - U_n^t\right) - \min\left(M, U_n^t, L - U_{n+1}^t\right) + U_n^t$$

☞ 従属変数の離散化: $M,\ L,\ U_n^0 \in \mathbb{Z} \quad \Longrightarrow \quad U_n^t \in \mathbb{Z}$

☞ セルオートマトン化: パラメータを調整して U_n^t の値を有限個に制限できる

$$M, L > 0, \quad 0 \le U_n^0 \le L \quad \longrightarrow \quad 0 \le U_n^t \le L$$

$(\because)$

$$\begin{cases} \bullet\ \min\left(M, U_{n-1}^0, L - U_n^0\right) \ge 0, \\ \bullet\ \min\left(M, U_n^0, L - U_{n+1}^0\right) \ge 0, \end{cases} \quad \begin{cases} \bullet\ \min\left(M, U_{n-1}^0, L - U_n^0\right) + U_n^0 = \min\left(M + U_n^0, U_{n-1}^0 + U_n^0, L\right) \le L \\ \bullet\ \min\left(M, U_n^0, L - U_{n+1}^0\right) - U_n^0 = \min\left(M - U_n^0, 0, L - U_{n+1}^0 - U_n^0\right) \le 0 \end{cases}$$

☞ **2値セルオートマトンへの簡約化:**

$$L < M \quad \longrightarrow \quad U_n^{t+1} = \min\left(U_{n-1}^t, L - U_n^t\right) - \min\left(U_n^t, L - U_{n+1}^t\right) + U_n^t$$

$L = 1$: **Rule 184 Elementary Cellular Automaton (Wolfram)**

$$\frac{U_{n-1}^t\ U_n^t\ U_{n+1}^t}{U_n^t} : \quad \frac{111}{1} \quad \frac{110}{0} \quad \frac{101}{1} \quad \frac{100}{1} \quad \frac{011}{1} \quad \frac{010}{0} \quad \frac{001}{0} \quad \frac{000}{0} \qquad 10111000_{(2)} = 184$$

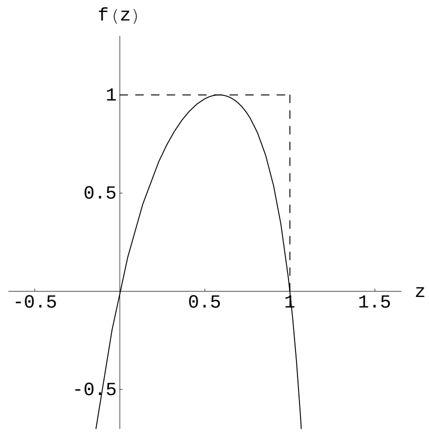
Schröder map の超離散化 (1)

Schröder map: × 「マイナスの問題」

$$z_{n+1} = f(z_n) = \frac{4z_n(1-z_n)(1-k^2 z_n)}{(1-k^2 z_n^2)^2},$$

$$z_n = \mathrm{sn}^2(2^n u_0; k) \in [0,1]$$

$$\lim_{\epsilon\to+0} \epsilon \log\left(e^{\frac{A}{\epsilon}} + e^{\frac{B}{\epsilon}} + \cdots\right) = \max(A, B, \cdots)$$



log 内の各項は正の実数でなければならない！

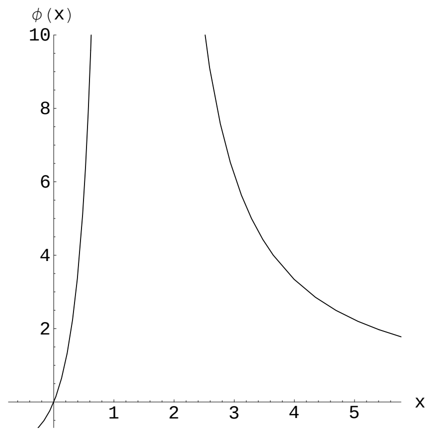
✓ポイント：[0, 1] は超離散化にとって制限がきつすぎる ⟶ 適当な一次分数変換で $[0,\infty)$ に写す

$$z_n \longmapsto x_n = \frac{z_n}{1-z_n}$$

Schröder map (on $[0,\infty)$): 超離散化可能！

$$x_{n+1} = \varphi(x_n) = \frac{4x_n(1+x_n)\left(1+k'^2 x_n\right)}{(1-k'^2 x_n^2)^2}$$

$$x_n = \frac{z_n}{1-z_n} = \frac{\mathrm{sn}^2(2^n u_0;k)}{\mathrm{cn}^2(2^n u_0;k)}, \quad k'^2 = 1-k^2$$



Schröder map の超離散化 (2)

✓ 注意： 写像の対応： $z \to -x$, $k \to k'$ 解の対応： **Jacobi** の虚数変換 $\dfrac{\mathrm{sn}(u;k)}{\mathrm{cn}(u;k)} = -i\,\mathrm{sn}(iu;k')$

$$z_{n+1} = \frac{4z_n(1-z_n)(1-k^2z_n)}{(1-k^2z_n^2)^2}, \quad z_n = \mathrm{sn}^2(2^nu_0;k) \quad \longrightarrow \quad x_{n+1} = \frac{4x_n(1+x_n)\left(1+k'^2x_n\right)}{(1-k'^2x_n^2)^2}, \quad x_n = \frac{\mathrm{sn}^2(2^nu_0;k)}{\mathrm{cn}^2(2^nu_0;k)}$$

✓ 超離散化： $x_n = \exp\left[\dfrac{X_n}{\epsilon}\right]$, $k' = \exp\left[-\dfrac{L}{2\epsilon}\right]$, $(0 < k' < 1,\ L > 0)$.

Ultradiscrete Schröder map

$$X_{n+1} = \lim_{\epsilon \to +0} \epsilon \log\left[\frac{4e^{\frac{X_n}{\epsilon}}(1+e^{\frac{X_n}{\epsilon}})(1+e^{\frac{X_n-L}{\epsilon}})}{(1-e^{\frac{2X_n-L}{\epsilon}})^2}\right]$$

$$= X_n + \max(0,X_n) + \max(0,X_n-L) - 2\max(0,2X_n-L) = \begin{cases} X_n & X_n < 0 \\ 2X_n & 0 \le X_n < \frac{L}{2} \\ -2X_n + 2L & \frac{L}{2} \le X_n < L \\ -X_n + L & L \le X_n \end{cases}$$

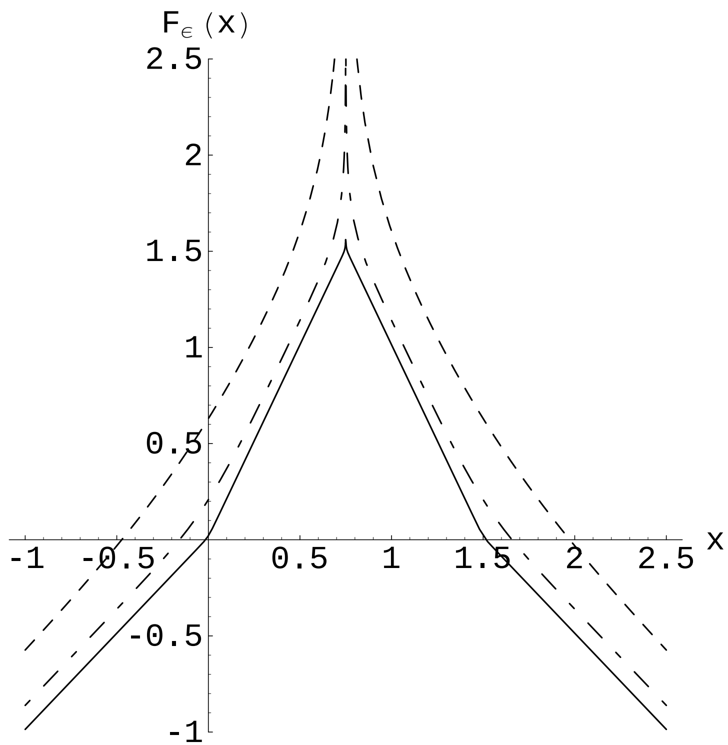
☞ 注意：log の中にも **dominant** にならない限り負号を含んでよい

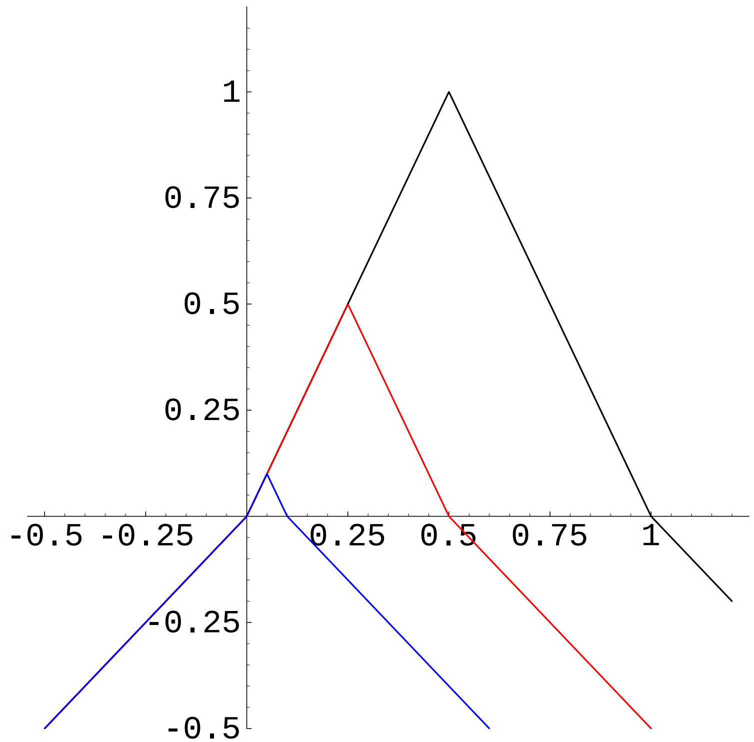
$$\lim_{\epsilon \to +0} \epsilon \log\left(e^{\frac{A}{\epsilon}} - e^{\frac{B}{\epsilon}}\right)^2 = \lim_{\epsilon \to +0} \epsilon \log\left(e^{\frac{2A}{\epsilon}} - 2e^{\frac{A+B}{\epsilon}} + e^{\frac{2B}{\epsilon}}\right) = 2\max(A,B)$$

Schröder map の超離散化 (3)

Ultradiscrete Schröder map

$$X_{n+1} = X_n + \max(0, X_n) + \max(0, X_n - L) - 2\max(0, 2X_n - L) = \begin{cases} X_n & X_n < 0 \\ 2X_n & 0 \le X_n < \frac{L}{2} \\ -2X_n + 2L & \frac{L}{2} \le X_n < L \\ -X_n + L & L \le X_n \end{cases}$$





✓ $[0, L]$ では **tent map**，その外では自明な軌道

Schröder map の超離散化 (4): 解の超離散化

楕円テータ函数

$$\mathrm{sn}(u;k)=\frac{\vartheta_3(0)\vartheta_1(v)}{\vartheta_2(0)\vartheta_0(v)},\quad \mathrm{cn}(u;k)=\frac{\vartheta_0(0)\vartheta_2(v)}{\vartheta_2(0)\vartheta_0(v)},\quad u=\pi(\vartheta_3(0))^2 v,\quad k^2=\left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4,$$

$$\vartheta_0(v)=\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}z^{2n}=\sqrt{\frac{\theta}{\epsilon\pi}}\sum_{n\in\mathbb{Z}}e^{-\frac{\theta}{\epsilon}[v-(n+\frac{1}{2})]^2}\qquad \vartheta_2(v)=\sum_{n\in\mathbb{Z}}q^{(n-\frac{1}{2})^2}z^{2n-1}=\sqrt{\frac{\theta}{\epsilon\pi}}\sum_{n\in\mathbb{Z}}(-1)^n e^{-\frac{\theta}{\epsilon}(v-n)^2}$$

$$\vartheta_1(v)=i\sum_{n\in\mathbb{Z}}(-1)^n q^{(n-\frac{1}{2})^2}z^{2n-1}=\sqrt{\frac{\theta}{\epsilon\pi}}\sum_{n\in\mathbb{Z}}(-1)^n e^{-\frac{\theta}{\epsilon}[v-(n+\frac{1}{2})]^2}\qquad \vartheta_3(v)=\sum_{n\in\mathbb{Z}}q^{n^2}z^{2n}=\sqrt{\frac{\theta}{\epsilon\pi}}\sum_{n\in\mathbb{Z}}e^{-\frac{\theta}{\epsilon}(v-n)^2}$$

$$z=\exp[i\pi v],\quad q=\exp\left[-\frac{\epsilon\pi^2}{\theta}\right],\quad \tau=i\frac{\epsilon\pi}{\theta},\quad \theta>0$$

$\epsilon\sim+0$での漸近挙動 $\qquad ((v))=v-\mathrm{Floor}(v),\quad 0\le((v))<1$

✓ $\vartheta_0(0)\sim 2\sqrt{\frac{\theta}{\epsilon\pi}}\exp\left[-\frac{\theta}{4\epsilon}\right]$

✓ $(\vartheta_0(v))^2\sim\left(\frac{\theta}{\epsilon\pi}\right)\exp\left[-\frac{2\theta}{\epsilon}[((v))-\frac{1}{2}]^2\right]$

✓ $\vartheta_2(0)\sim\sqrt{\frac{\theta}{\epsilon\pi}}\left(1-2\exp\left[-\frac{\theta}{\epsilon}\right]\right)$

✓ $(\vartheta_1(v))^2\sim\left(\frac{\theta}{\epsilon\pi}\right)\exp\left[-\frac{2\theta}{\epsilon}[((v))-\frac{1}{2}]^2\right]$

✓ $\vartheta_3(0)\sim\sqrt{\frac{\theta}{\epsilon\pi}}\left(1+2\exp\left[-\frac{\theta}{\epsilon}\right]\right)$

✓ $(\vartheta_2(v))^2\sim\left(\frac{\theta}{\epsilon\pi}\right)\left(\exp\left[-\frac{\theta}{\epsilon}[((v))]^2\right]-\exp\left[-\frac{\theta}{\epsilon}[((v))-1]^2\right]\right)^2$

Schröder map の超離散化 (5): 解の超離散化

$\epsilon \sim +0$での漸近挙動　　$((\nu)) = \nu - \mathrm{Floor}(\nu), \quad 0 \le ((\nu)) < 1$

✓ $\vartheta_0(0) \sim 2\sqrt{\frac{\theta}{\epsilon\pi}} \exp\left[-\frac{\theta}{4\epsilon}\right]$

✓ $\vartheta_2(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 - 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)$

✓ $\vartheta_3(0) \sim \sqrt{\frac{\theta}{\epsilon\pi}} \left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)$

✓ $(\vartheta_0(\nu))^2 \sim \left(\frac{\theta}{\epsilon\pi}\right) \exp\left[-\frac{2\theta}{\epsilon}[((\nu)) - \frac{1}{2}]^2\right]$

✓ $(\vartheta_1(\nu))^2 \sim \left(\frac{\theta}{\epsilon\pi}\right) \exp\left[-\frac{2\theta}{\epsilon}[((\nu)) - \frac{1}{2}]^2\right]$

✓ $(\vartheta_2(\nu))^2 \sim \left(\frac{\theta}{\epsilon\pi}\right) \left(\exp\left[-\frac{\theta}{\epsilon}[((\nu))]^2\right] - \exp\left[-\frac{\theta}{\epsilon}[((\nu)) - 1]^2\right]\right)^2$

$$\Downarrow$$

$$k'^2 = \exp\left[-\frac{L}{2\epsilon}\right] = 1 - k^2 = 1 - \left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4 = \frac{16\exp\left[-\frac{\theta}{\epsilon}\right]\left(1 + 4\exp\left[-\frac{2\theta}{\epsilon}\right]\right)}{\left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)^4}$$

$$x_n = \exp\left[\frac{X_n}{\epsilon}\right] = \frac{\mathrm{sn}^2(u;k)}{\mathrm{cn}^2(u;k)} = \left(\frac{\vartheta_3(0)\vartheta_1(\nu)}{\vartheta_0(0)\vartheta_2(\nu)}\right)^2 = \frac{\left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)^2 \exp\left[\frac{2\theta((\nu))}{\epsilon}\right]}{4\left(1 - \exp\left[\frac{\theta}{\epsilon}[2((\nu)) - 1]\right]\right)^2}$$

Schröder map の超離散化 (6): 解の超離散化

$$L = \lim_{\epsilon \to +0} \epsilon \left[\frac{\theta}{\epsilon} - \log\left(1 + 4\exp\left[-\frac{2\theta}{\epsilon}\right]\right) + \log\left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)^4 \right] = \theta - \max[0, -2\theta] + 4\max[0, -\theta] = \theta$$

$$X_n = \lim_{\epsilon \to +0} \epsilon \left[\log\left(1 + 2\exp\left[-\frac{\theta}{\epsilon}\right]\right)^2 + \frac{2\theta((\nu))}{\epsilon} - \log\left(1 - \exp\left[\frac{\theta}{\epsilon}[2((\nu)) - 1]\right]\right)^2 \right]$$

$$= 2\left(\max[0, -\theta] + \theta((\nu)) - \max[0, \theta(2((\nu)) - 1)]\right) = \theta\left(1 - 2\left|\,((\nu)) - \frac{1}{2}\,\right|\right), \quad \nu = 2^n \nu_0,$$

以上より，**Schröder map** とその解の超離散化は次のようになる：

Ultradiscrete Schröder map

$$X_{n+1} = L\left(1 - 2\left|\,\frac{X_n}{L} - \frac{1}{2}\,\right|\right), \quad X_n \in [0, L]$$

Solution:

$$X_n = L\left(1 - 2\left|\,((2^n \nu_0)) - \frac{1}{2}\,\right|\right)$$

✓ 当然，この解が実際に解であることは直接計算でも示される．

Schröder map の超離散化 (7): いくつかの注意

☞ **解の引数のparametrization:**

解の引数の中の任意定数 u_0, v_0 は次のようにスケールを取る必要がある： $u_0 = \dfrac{\theta}{\epsilon} v_0$

こうしておくと

$$v = \frac{u}{\pi(\vartheta_3(0))^2} \approx \frac{2^n u_0}{\pi\left(\frac{\theta}{\epsilon\pi}\right)} \longrightarrow 2^n v_0 \quad (\epsilon \to +0)$$

☞ **超離散極限と楕円函数の周期:** $\frac{\mathrm{sn}^2(u;k)}{\mathrm{cn}^2(u;k)}$ の基本周期は $2K(k)$, $2iK(k')$

$$\epsilon \to +0: \quad K(k) = \frac{\pi}{2}(\vartheta_3(0))^2 \approx \frac{\theta}{2\epsilon}, \quad K(k') = -\frac{\pi i \tau}{2}(\vartheta_3(0))^2 \approx \frac{\pi}{2}, \quad u \approx \frac{\theta}{\epsilon} v$$

$\longrightarrow$ v に関する基本周期 $\approx 1, \quad \dfrac{i\epsilon\pi}{\theta}$

超離散化は実周期を有限に保ったまま，虚周期を0にすることで実現されている！

☞ **Logistic map:** $k = 0 \longleftrightarrow L = 0$

$$z_{n+1} = 4z_n(1 - z_n) \longrightarrow x_{n+1} = \frac{4x_n}{(1 - x_n)^2} \longrightarrow X_{n+1} = -|X_n|, \quad X_n = 0 :$$ 面白くない

$$k \to 0: \quad K(k) \approx \frac{\pi}{2}, \quad K(k') \approx \log\frac{4}{k}$$ 超離散極限と**consistent**でない!

Schröder map に付随する楕円曲線

$[0,\infty)$ 上の **Schröder map:**

$$x_{n+1} = \frac{4x_n(1+x_n)\left(1+k'^2 x_n\right)}{(1-k'^2 x_n^2)^2}, \quad k' = \sqrt{1-k^2}, \quad x_n = \frac{\mathrm{sn}^2(2^n\xi_0;k)}{\mathrm{cn}^2(2^n\xi_0;k)}$$

✓ ある楕円曲線上の点の運動の x 軸上への射影と見なせる

$$[xy - a(x+y) + b]^2 = 4d^2 xy, \quad (x,y) = \left(\frac{\mathrm{sn}^2(u;k)}{\mathrm{cn}^2(u;k)}, \frac{\mathrm{sn}^2(u+\eta;k)}{\mathrm{cn}^2(u+\eta;k)}\right)$$

$$a = \frac{1}{k'^2}, \quad b = \frac{1}{k'^2}\frac{\mathrm{cn}^2(\eta;k)}{\mathrm{sn}^2(\eta;k)}, \quad d = -\frac{1}{k'^2}\frac{\mathrm{dn}(\eta;k)}{\mathrm{sn}^2(\eta;k)} \longrightarrow k'^2 d^2 = (1+k'^2 b)(1+b)$$

✓ 楕円曲線の超離散化：

$$x = e^{\frac{X}{\epsilon}}, \quad y = e^{\frac{Y}{\epsilon}}, \quad b^2 = e^{\frac{B}{\epsilon}}, \quad 4d^2 = e^{\frac{D}{\epsilon}}, \quad k' = e^{-\frac{L}{2\epsilon}}\ (L>0) \quad a = \frac{1}{k'^2} = e^{\frac{L}{\epsilon}}, \quad \epsilon \to +0$$

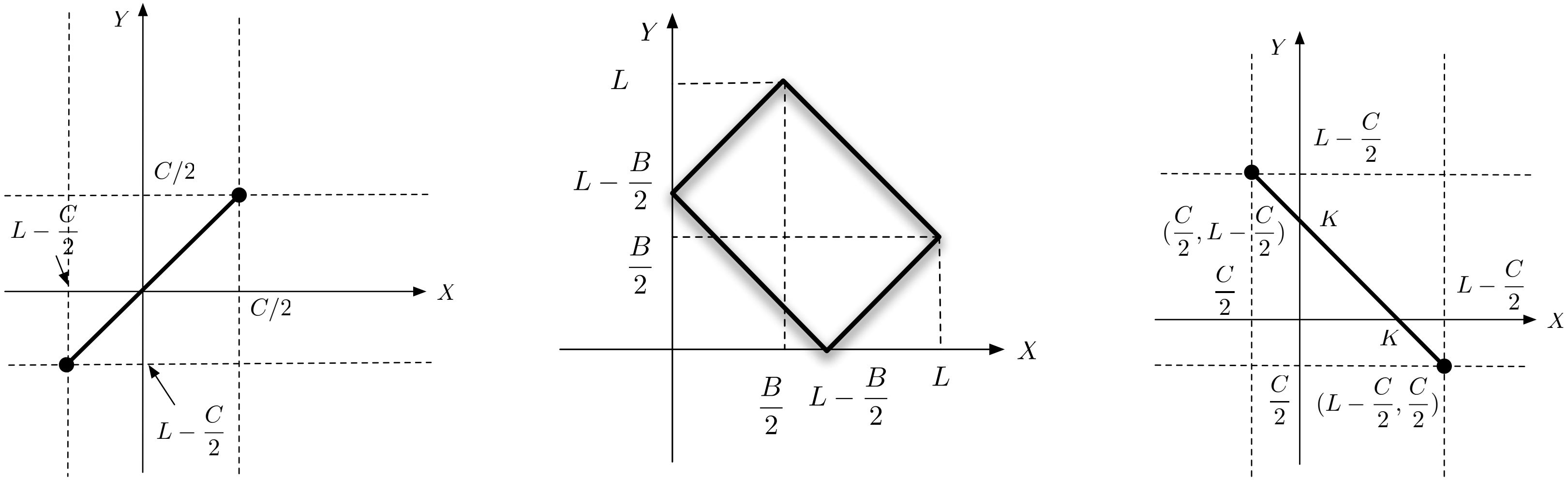
$\longrightarrow$

- 超離散化された楕円曲線: $\max(2X+2Y, B+2X, B+2Y, 2L) = X+Y+D$
- パラメータの条件: $-L + D = \max\left(0, \frac{B}{2} - L\right) + \max\left(0, \frac{B}{2}\right)$

楕円曲線の超離散化(1)

$$\max(2X+2Y, B+2X, B+2Y, 2L) = X+Y+D, \quad -L+D = \max\left(0, \frac{B}{2}-L\right) + \max\left(0, \frac{B}{2}\right)$$

1. $B > 2L\ (>0)$ の場合： $-L+D = \frac{B}{2} - L + \frac{B}{2} \implies B = D\ (>0)$
2. $2L > B > 0$ の場合： $-L+D = \frac{B}{2} \implies D = L + \frac{B}{2}\ (>0)$
3. $0 > B$ の場合： $-L+D = 0 \implies D = L\ (>0)$



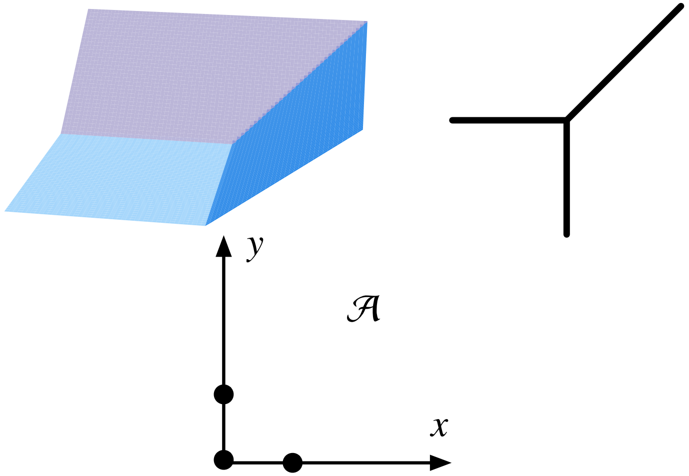
以後 2: $2L > C > 0, \quad D = L + \frac{C}{2}$ の場合のみ考える

トロピカル曲線(1)

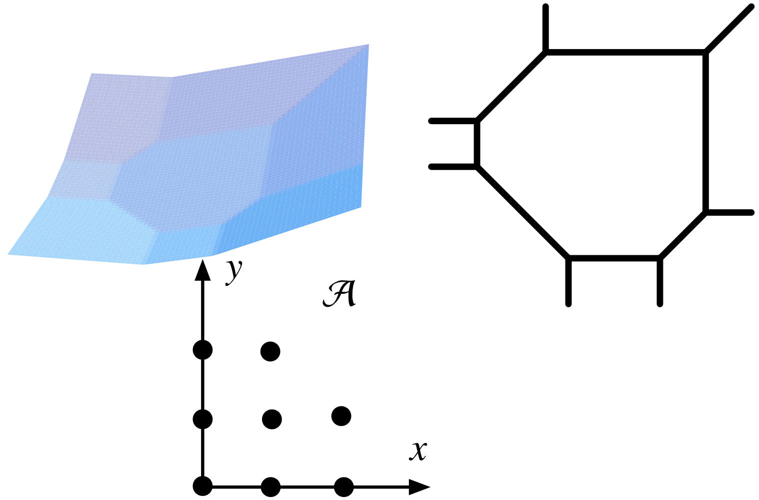
✓　トロピカル多項式 f から定まるトロピカル曲線 C： f が微分不可能となる点の集合

$$f = \max_{(a_1,a_2)\in\mathcal{A}} \left[\lambda_{(a_1,a_2)} + a_1 x + a_2 y\right], \quad \mathcal{A} \in \mathbb{Z}^2$$

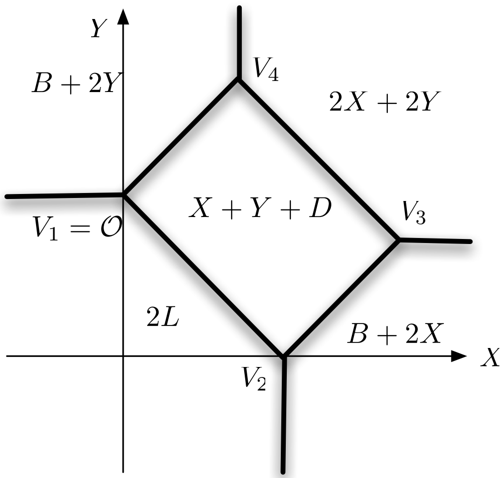
トロピカル直線



トロピカル曲線



$$f = \max(2X + 2Y, B + 2X, B + 2Y, 2L, X + Y + D), \quad 2L > B > 0, \quad D = L + \frac{B}{2}$$



前ページで得られた曲線は C から半直線 **(tentacles)** を取り去ったもの: $\overline{C}$

トロピカル楕円曲線 (1)

✓ トロピカル楕円曲線 (**M.D. Vigeland, math.AG/0411485**): $f = \max_{(a_1,a_2)\in\mathcal{A}} [\lambda_{(a_1,a_2)} + a_1 x + a_2 y]$

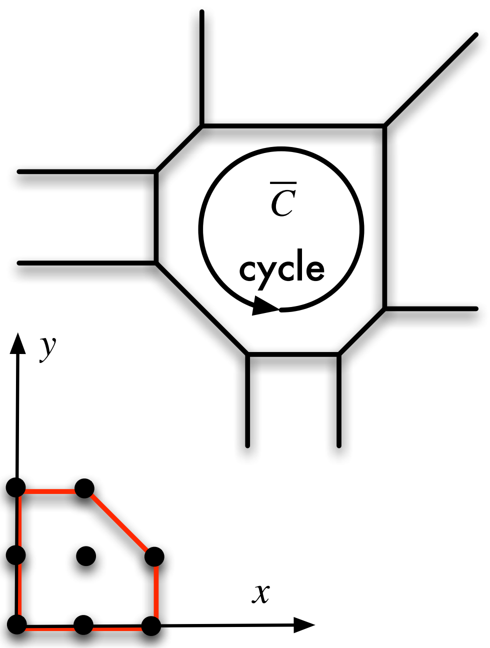
☞ **degree 3**: $\mathcal{A} = \{(a_1, a_2) \in \mathbb{Z}^2 \mid 0 \leq a_1, a_2 \leq 2, 0 \leq a_1 + a_2 \leq 3\}$

☞ **genus 1** ($\mathcal{A}$ の凸包 $\Delta(\mathcal{A})$ の内部の格子点の数)

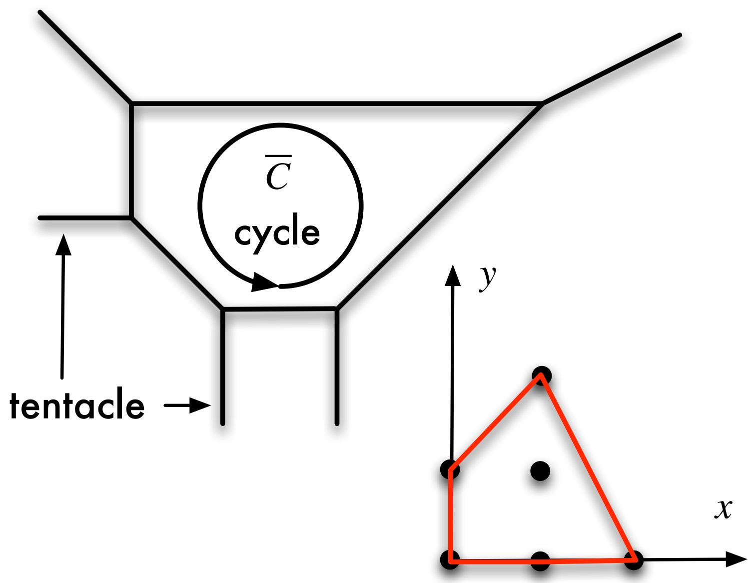
☞ 滑らか

C: 一つのサイクル $\overline{C}$ と各頂点から出る半直線 (**tentacle**)

$$\lambda_{(2,0)} = \lambda_{(0,2)} = \lambda_{(0,0)} = 0$$
$$\lambda_{(2,1)} = \lambda_{(1,2)} = 10$$
$$\lambda_{(1,0)} = \lambda_{(0,1)} = 5 \quad \lambda_{(1,1)} = 20$$



$$\lambda_{(2,0)} = \lambda_{(0,0)} = 0$$
$$\lambda_{(1,2)} = 10$$
$$\lambda_{(1,0)} = \lambda_{(0,1)} = 5$$
$$\lambda_{(1,1)} = 20 \quad \lambda_{(0,2)} = \lambda_{(2,1)} = -\infty$$



トロピカル楕円曲線 (2)

✓ 「滑らか」なトロピカル曲線

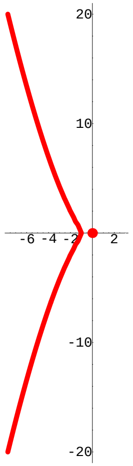
☞ 全ての頂点が三叉

☞ 各頂点から出る辺の **primitive tangent vector** の和がゼロ.

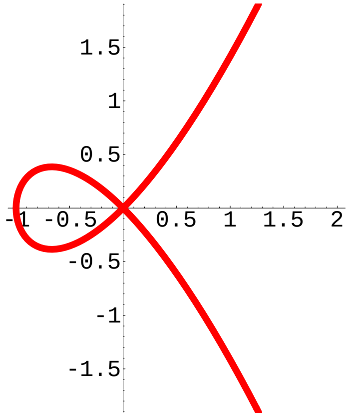
※ (p, q) がある辺の **primitive tangent vector** $\longleftrightarrow$ **(1)** 辺に平行 **(2)** p, q は互いに素な整数

滑らかでない場合

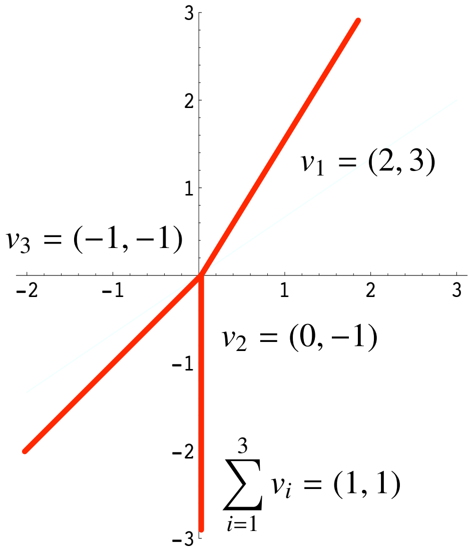
$y^2 = -x^3 - x^2$



$y^2 = x^3 + x^2$

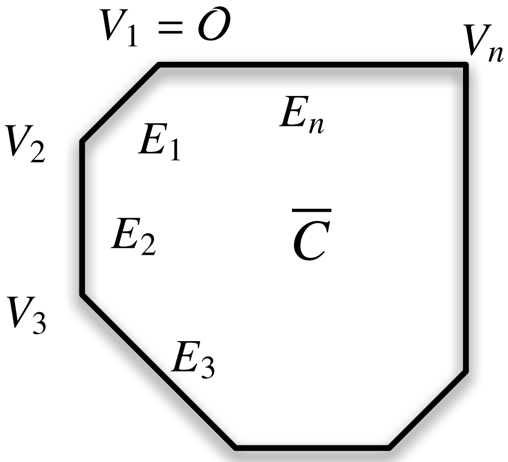


$\max[2y, 3x, 2x]$



トロピカル楕円曲線 (3): Jacobi 多様体 と Abel-Jacobi map

Mikhalkin-Zharkov(2006), Inoue-Takenawa (2008)



$\varepsilon_i = \dfrac{1}{|\boldsymbol{v}_i|}$ ($\boldsymbol{v}_i$: E_i の primitive tangent vector)

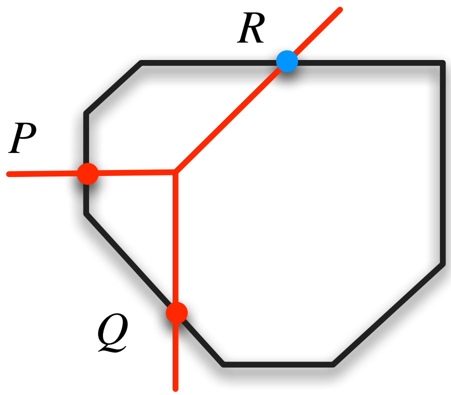
Total lattice length: $\mathcal{L} = \sum_{i=1}^{n} \varepsilon_i |E_i|$

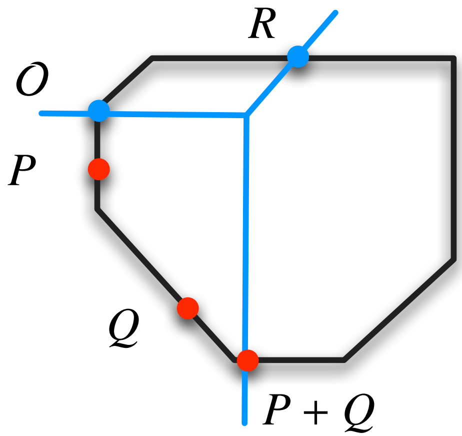
tropical Jacobian: $J(\overline{C}) = \mathbb{R}/\mathcal{L}\mathbb{Z}$

✓ **Abel-Jacobi map:**

$\eta : \overline{C} \longrightarrow J(\overline{C})$ 区分線形写像(E_i 上で線形) $\eta(O) = 0,\quad \eta(V_{i+1}) = \eta(V_i) + \varepsilon_i|E_i| \quad (i = 1, 2, \ldots, n-1)$

✓ トロピカル楕円曲線の加法 **(Vigeland, 2004)**

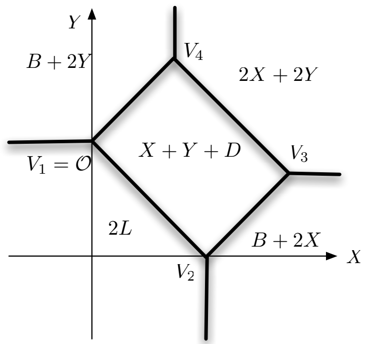


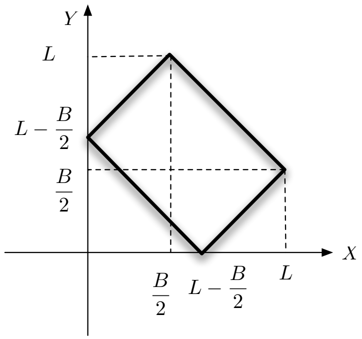


ultradiscrete Schröder map と トロピカル曲線

$f = \max(2X + 2Y, B + 2X, B + 2Y, 2L, X + Y + D)$
の定めるトロピカル曲線 C

$\overline{C}$: $\max(2X + 2Y, B + 2X, B + 2Y, 2L) = X + Y + D$
Schröder map に付随する楕円曲線の超離散化





✓ Tropical Jacobian と Abel-Jacobi map

$\overline{C}$ の頂点： $V_1 = \mathcal{O} = \left(0, L - \frac{B}{2}\right), \quad V_2 = \left(L - \frac{B}{2}, 0\right), \quad V_3 = \left(L, \frac{B}{2}\right), \quad V_4 = \left(\frac{B}{2}, L\right)$

$\overline{C}$ の辺長: $E_i = V_iV_{i+1}, \quad |E_1| = \sqrt{2}\left(L - \frac{B}{2}\right), \quad |E_2| = \frac{\sqrt{2}}{2}B, \quad |E_3| = \sqrt{2}\left(L - \frac{B}{2}\right), \quad |E_4| = \frac{\sqrt{2}}{2}B$

primitive tangent vector： $\boldsymbol{v}_1 = (1, -1), \quad \boldsymbol{v}_2 = (1, 1), \quad \boldsymbol{v}_3 = (-1, 1), \quad \boldsymbol{v}_4 = (-1, -1)$

Total lattice length： $\mathcal{L} = \sum_{i=1}^{4} \frac{|E_i|}{|\boldsymbol{v}_i|} = 2L \longrightarrow$ **Tropical Jacobian**： $J(\overline{C}) = \mathbb{R}/\mathcal{L}\mathbb{Z} = \mathbb{R}/2L\mathbb{Z}$

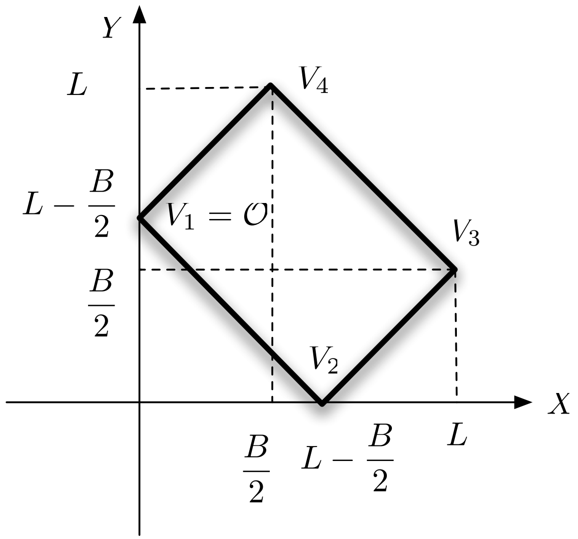
$\Longrightarrow$ **Abel-Jacobi map** $\mu : \overline{C} \longrightarrow J(\overline{C}), \quad \mu(V_1) = 0, \quad \mu(V_2) = L - \frac{B}{2}, \quad \mu(V_3) = L, \quad \mu(V_4) = 2L - \frac{B}{2}$

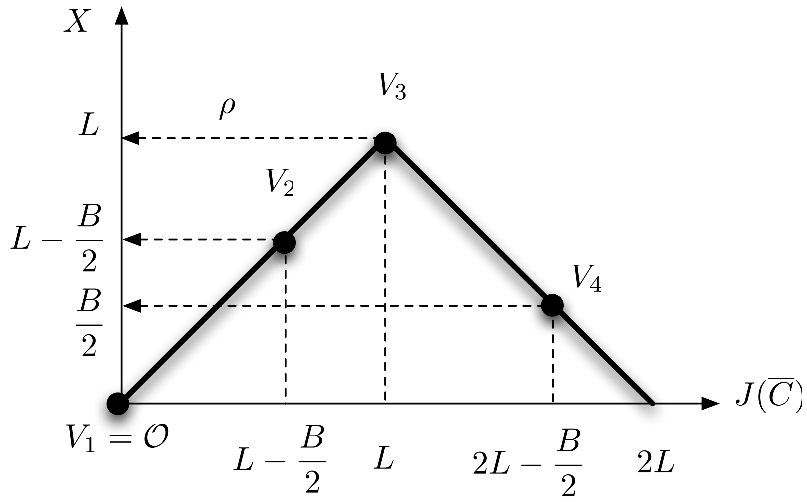
Tropical Jacobian 上の倍角写像 (1)

✓ $\pi: \overline{C} \longrightarrow \mathbb{R}$: $\overline{C}$ 上の点から X 軸への射影.

ただし，π^{-1} は **1:2** 写像であるので，Y 座標の小さい方を取ると約束する.

✓ $\rho = \pi \circ \mu^{-1}: J(\overline{C}) \longrightarrow \mathbb{R}$: $P \in \overline{C}$ の X 座標と $\mu(P)$ を対応づける写像





✓ **Tropical Jacobian 上の倍角写像**

$$\varphi_2: J(\overline{C}) \to J(\overline{C}), \qquad \varphi_2(p) \equiv 2p \ (\mathrm{mod}\ \mathcal{L}), \quad p \in J(\overline{C})$$

✓ ρ による φ_2 の共役写像

$$\Phi_2: \mathbb{R} \to \mathbb{R}, \qquad \Phi_2 = \rho \circ \varphi_2 \circ \rho^{-1}$$

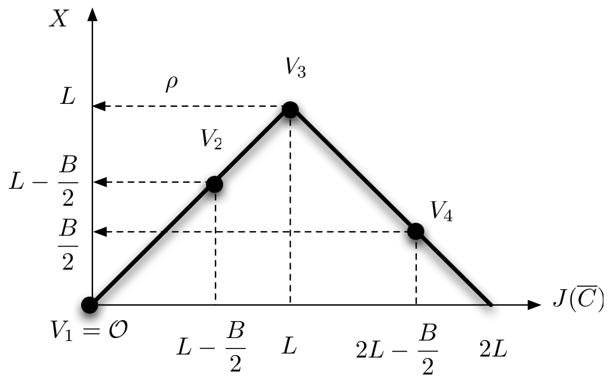
✓ 主張したいこと：

$$X_{n+1} = \Phi_2(X_n): \quad [0, L] \text{上の } \textbf{ultradiscrete Schröder map (tent map)}$$

Tropical Jacobian 上の倍角写像 (2)

✓ $\rho = \mu \circ \pi^{-1}$ による$\overline{C}$のX 座標と $J(\overline{C})$ の **explicit** な対応：

$$\rho(p) = \begin{cases} p & 0 \le p \le L \\ -p + 2L & L \le p \le 2L \end{cases} \qquad p \in J(\overline{C})$$



✓ $\Phi_2 = \rho \circ \varphi_2 \circ \rho^{-1}$ を書き下す：

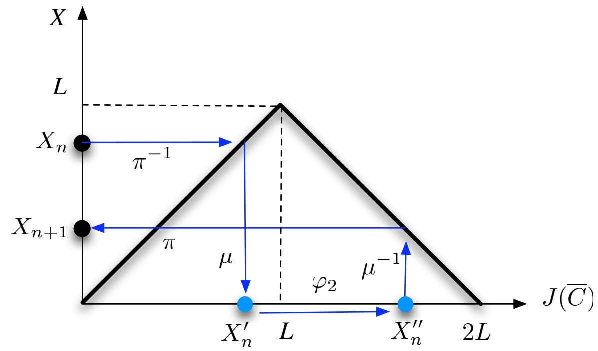
$$X' := \rho^{-1}(X) = X, \quad X'' := \varphi_2(X') = 2X' = 2X, \quad X', X'' \in J(\overline{C})$$

☞ $0 \le X \le \frac{L}{2}$ のとき：

$$0 \le X'' \le L \longrightarrow \Phi_2(X) = \rho(X'') = 2X$$

☞ $\frac{L}{2} \le X \le L$ のとき：

$$L \le X'' \le 2L \longrightarrow \Phi_2(X) = \rho(X'') = -2X + 2L$$



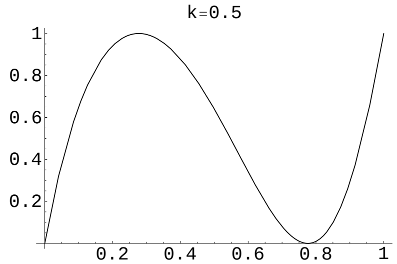
[0, L] 上の **ultradiscrete Schröder map**：

$$X_{n+1} = \Phi_2(X_n) = L\left(1 - \left|\frac{X_n}{L} - \frac{1}{2}\right|\right) = \begin{cases} 2X_n & 0 \le X \le \frac{L}{2} \\ -2X_n + 2L & \frac{L}{2} \le X \le L \end{cases}$$

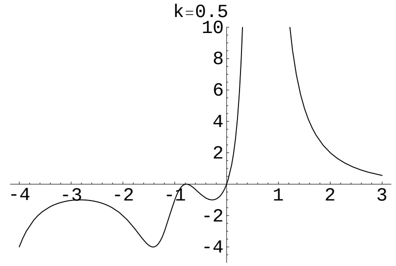
Generalized cubic map (1)

✓ **"Generalized cubic map"**

$$z_{n+1} = \frac{z_n\left[k^4 z_n^4 - 6k^2 z_n^2 + 4(k^2+1)z_n - 3\right]^2}{\left[3k^4 z_n^4 - 4k^2(k^2+1)z_n^3 + 6k^2 z_n^2 - 1\right]^2}, \qquad z_n = \mathrm{sn}^2(3^n u_0; k)$$



$$x_{n+1} = \frac{x_n\left[k'^4 x_n^4 - 6k'^2 x_n^2 - 4(k'^2+1)x_n - 3\right]^2}{\left[3k'^4 x_n^4 + 4k'^2(k'^2+1)x_n^3 + 6k'^2 x_n^2 - 1\right]^2}, \qquad x_n = \frac{\mathrm{sn}^2(3^n u_0; k)}{\mathrm{cn}^2(3^n u_0; k)}$$



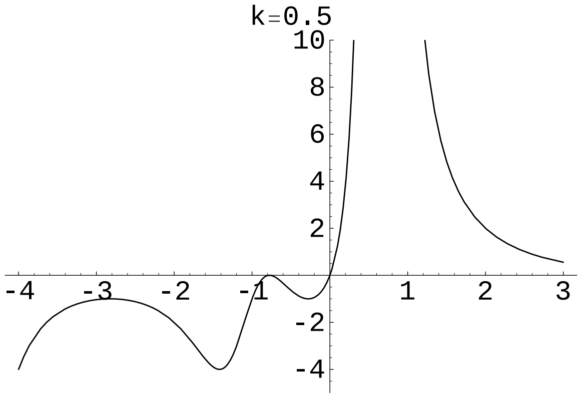
$$x_n = \frac{z_n}{1-z_n}, \quad k'^2 = 1 - k^2, \qquad x_n = \exp\left[\frac{X_n}{\epsilon}\right], \quad k' = \exp\left[-\frac{L}{2\epsilon}\right] \quad (L > 0)$$

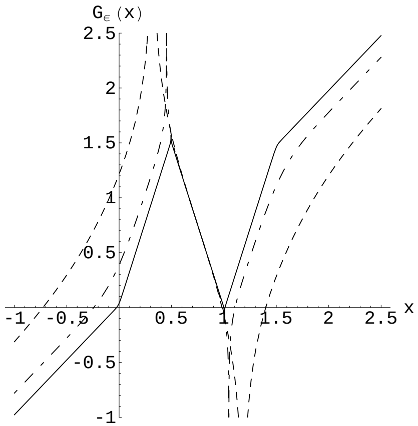
ultra-discrete generalized cubic map

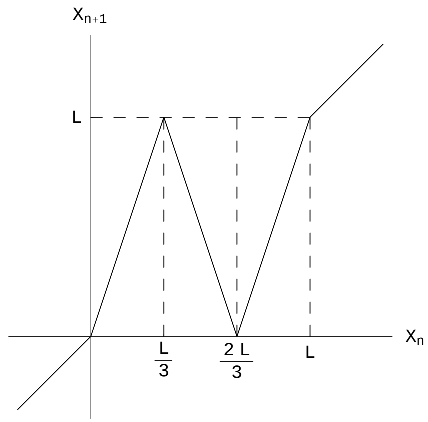
$$X_{n+1} = X_n + 2\max(0, X_n, 4X_n - 2L) - 2\max(0, 3X_n - L, 4X_n - 2L) = \begin{cases} X_n & X_n < 0 \\ 3X_n & 0 \le X_n < \frac{L}{3} \\ -3X_n + 2L & \frac{L}{3} \le X_n < \frac{2L}{3} \\ 3X_n - 2L & \frac{2L}{3} \le X_n < L \\ X_n & L \le X_n \end{cases}$$

Generalized cubic map (2)

$$X_{n+1} = X_n + 2\max(0, X_n, 4X_n - 2L) - 2\max(0, 3X_n - K, 4X_n - 2L), \quad X_n = L\left(1 - 2\left|\, ((3^n \nu_0)) - \frac{1}{2}\,\right|\right)$$





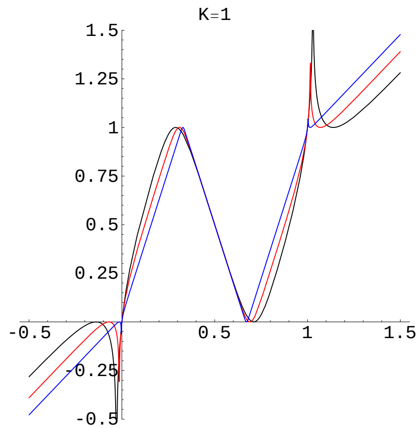


✓ 注意： z_n に関する写像を直接超離散化： 写像函数の極限: ○, 解の極限: ×

$$z_{n+1} = \frac{z_n\left[k^4 z_n^4 - 6k^2 z_n^2 + 4(k^2+1)z_n - 3\right]^2}{\left[3k^4 z_n^4 - 4k^2(k^2+1)z_n^3 + 6k^2 z_n^2 - 1\right]^2}$$

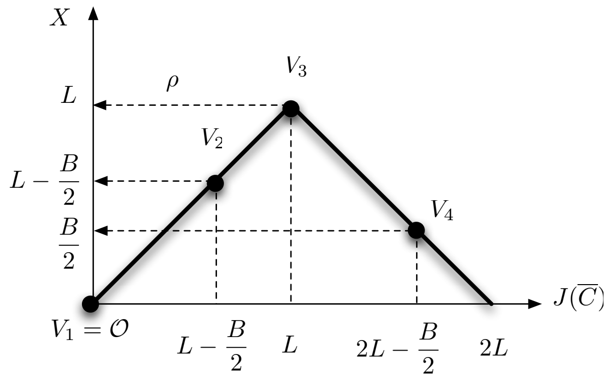
$$z_n = \exp\left[\frac{Z_n}{\epsilon}\right], \quad k = \exp\left[-\frac{L}{2\epsilon}\right] \quad (L > 0)$$

$$Z_n = \lim_{\epsilon \to +0} \epsilon \log \mathrm{sn}^2(3^n u_0; k) = 0$$



Tropical Jacobian 上の3倍角写像

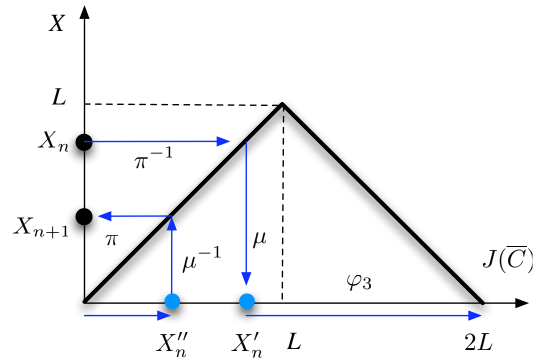
✓ $\rho(p) = \begin{cases} p & 0 \le p \le L \\ -p + 2L & L \le p \le 2L \end{cases} \quad p \in J(\overline{C})$



✓ $\Phi_3 = \rho \circ \varphi_3 \circ \rho^{-1}$, $\quad \varphi_3(p) \equiv 3p \bmod 2L$ を書き下す：

$$X' := \rho^{-1}(X) = X, \quad X'' := \varphi_3(X') = 3X' = 3X, \quad X', X'' \in J(\overline{C})$$

☞ $0 \le X \le \frac{L}{3}$： $\quad 0 \le X'' \le L \longrightarrow \Phi_3(X) = \rho(X'') = X'' = 3X$



☞ $\frac{L}{3} \le X \le \frac{2L}{3}$： $L \le X'' \le 2L \longrightarrow \Phi_3(X) = \rho(X'') = -X'' + 2L$
$$= -3X + 2L$$

☞ $\frac{2L}{3} \le X \le L$： $\quad 0 \le X'' \le L \longrightarrow \Phi_3(X) = \rho(X'') = X'' = 3X - 2L$

$[0, L]$ 上の **ultradiscrete generalized cubic map**：

$$X_{n+1} = \Phi_3(X_n) = \begin{cases} 3X_n & 0 \le X_n \le \frac{L}{3} \\ -3X_n + 2L & \frac{L}{3} \le X \le \frac{2L}{3} \\ 3X_n - 2L & \frac{2L}{3} \le X \le L \end{cases}$$

まとめ (1)

✓ **Schröder map** の超離散化

☛ 超離散化によって**Schröder map**の写像函数と解がセットで **tent map** に移行する.

$$z_{n+1} = \frac{4z_n(1-z_n)(1-k^2z_n)}{(1-k^2z_n^2)^2}, \quad z_n = \mathrm{sn}^2(2^n u_0; k)$$

$$\longrightarrow x_{n+1} = \frac{4x_n(1+x_n)\left(1+k'^2x_n\right)}{(1-k'^2x_n^2)^2}, \quad x_n = \frac{\mathrm{sn}^2(2^n u_0; k)}{\mathrm{cn}^2(2^n u_0; k)} = \left(\frac{\vartheta_3(0)\vartheta_1(\nu)}{\vartheta_0(0)\vartheta_2(\nu)}\right)^2$$

$$k^2 = \left(\frac{\vartheta_2(0)}{\vartheta_3(0)}\right)^4, \quad z = \exp[i\pi\nu], \quad u = \pi(\vartheta_3(0))^2\nu$$

$$x_n = \exp\left[\frac{X_n}{\epsilon}\right], \quad k' = \exp\left[-\frac{L}{2\epsilon}\right], \quad (0 < k' < 1,\ L > 0), \quad q = \exp\left[-\frac{\epsilon\pi^2}{\theta}\right], \quad \theta > 0$$

$\epsilon \to +0$: **Ultradiscrete Schröder map (tent map on** $[0, L]$**)**

$$X_{n+1} = \begin{cases} X_n & X_n < 0 \\ 2X_n & 0 \le X_n < \frac{L}{2} \\ -2X_n + 2L & \frac{L}{2} \le X_n < L \\ -X_n + L & L \le X_n \end{cases} \qquad X_n = \theta\left(1 - 2\left|\,((2^n\nu_0)) - \frac{1}{2}\,\right|\right), \quad \theta = L$$

☛ 超離散極限：虚周期 $\to 0$ とすることで実現

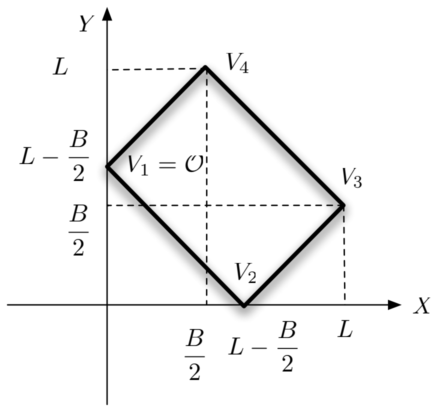
まとめ (2)

✓ **ultradiscrete Schröder map(とその一般化)の幾何学的解釈**

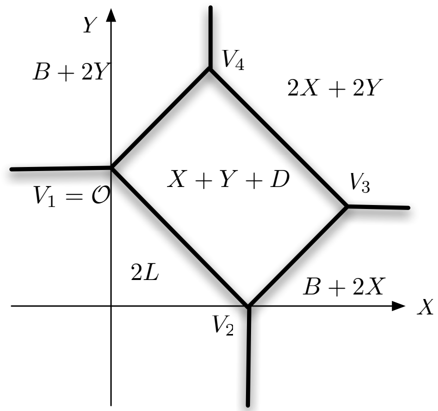
ある楕円曲線の超離散化から得られるトロピカル曲線の **Jacobian** 上の(*m*)倍角写像

$$[xy - c(x+y) + b]^2 = 4d^2xy, \quad (x, y) = \left(\frac{\mathrm{sn}^2(u;k)}{\mathrm{cn}^2(u;k)}, \frac{\mathrm{sn}^2(u+\eta;k)}{\mathrm{cn}^2(u+\eta;k)}\right)$$

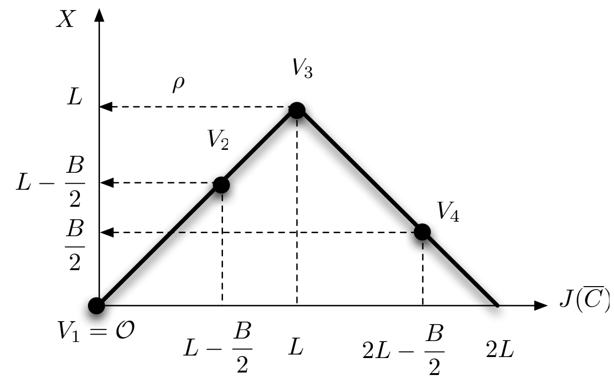
$\max(2X+2Y, C+2X, C+2Y, 2L)$
$= X + Y + D$



$C : \max(2X+2Y, C+2X, C+2Y,$
$2L, X+Y+D)$



Jacobian: $J(\overline{C}) = \mathbb{R}/2L\mathbb{Z}$



$\varphi_2(p) \equiv 2p \ (\mathrm{mod}\ 2L), \quad p \in J(\overline{C}):$ **Tropical Jacobian上の倍角写像**

$\rho = \pi \circ \mu^{-1} : J(\overline{C}) \longrightarrow \mathbb{R}$: $P \in \overline{C}$ の X 座標と $\mu(P)$ を対応づける写像

$X_{n+1} = \left(\rho \circ \varphi_2 \circ \rho^{-1}\right)(X_n)$: $[0, L]$ 上の **ultradiscrete Schröder map (tent map)**

参考文献

✓ 元となった論文

K. Kajiwara, A. Nobe and T. Tsuda, Ultradiscretization of one-dimensional solvable chaotic maps, arXiv:0803.4062v2

✓ 超離散化・**QRT**系・**Schröder map**

☞ 広田良吾，高橋大輔，「差分と超離散」，共立出版**(2003)**

☞ 中村佳正（編），「可積分系の応用数理」，裳華房**(2000)**

☞ A. Nobe, Ultradiscrete QRT maps and tropical elliptic curves, J. Phys. A: Math. Theor. **41**(2008) 125205

☞ T. Tsuda, Integrable mappings via rational elliptic surface, J. Phys. A: Math. Theor. **37**(2004) 2721-2730.

☞ E. Schröder, Ueber iterirte Functionen, Math. Ann. **3**(1871) 296-322.

☞ J. Milnor, On Lattès maps, math.DS/0402147v2.

✓ トロピカル幾何

☞ J. Richter-Gebert, B. Sturmfels and T. Theobald, First steps in tropical geometry, math.AG/0306366v2

☞ G. Mikhalkin, Enumerative tropical algebraic geometry in $\mathbb{R}^2$, math.AG/0312530v4

☞ M.D. Vigeland, The group law on a tropical elliptic curve, math.AG/0411485v1

☞ G. Mikhalkin and I. Zharkov, Tropical curves, their Jacobians and theta functions, math.AG/0612267v2

☞ R. Inoue and T. Takenawa, Tropical spectral curves and integrable cellular automata, Int. Math. Res. Notices **2008** article ID: rnn019