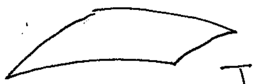
DISCRETE SURFACES

Plan: I) $\underbrace{\text{sine-Gordon vs.s sinh-Gordon}}_{\text{they've appeared in almost all the talks}}$, and surface theory. (Kajiwara, Kakei, Ohta)

II) isothermicity, Christoffel transforms, discrete minimal surfaces, discrete CMC 1 surfaces in $\mathbb{H}^3$.

III) Moebius geometry & discrete CMC surfaces in space forms, c.q.s, Calapso & $\underbrace{\text{Darboux \& Backlund}}_{\text{appeared especially in}}$ transforms. Kajiwara's, Wirrokusu's talks

I) 

$X = X(u,v) : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, N = unit normal

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle X_u, X_u \rangle & \\ \langle X_u, X_v \rangle & \langle X_v, X_v \rangle \end{pmatrix}, \quad II = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle X_{uu}, N \rangle & \\ \langle X_{uv}, N \rangle & \langle X_{vv}, N \rangle \end{pmatrix}$$

$K = \det(I^{-1} II)$, $H = \frac{1}{2} \cdot \operatorname{tr}(I^{-1} II)$, k_j = eigvals of $I^{-1} II$,

eigvec's = princ curvature directions $\longleftarrow \perp$, because I, II are symmetric

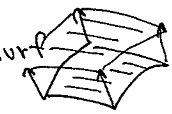
$K < 0$

$K > 0$

H depends on sign of N.

$H \equiv 0$, min. surf.
$H \equiv$ const, CMC H surf.

$K \equiv -1$, pseudospherical surface
$K \equiv 0$, flat surf
$K \equiv 1$, parallel to CMC surf



conformal coordinates: $\|X_u\| = \|X_v\|$, $X_u \perp X_v$ ("even" stretching, always exist, $E = G$ and $F = 0$, allow us to use Riemann surf theory and notion of holomorphicity on surfaces)

curvature line coordinates: X_u, X_v are princ curv directions ($F = f = 0$, always exist away from umbilics $k_1 = k_2$)

isothermic coord.s: conf'l and curv line ($\neq$ isothermal, don't always exist {Dini surf}, but surf's of rev and CMC surf's have them)

asymptotic line coord.s: X_u, X_v are asymptotic directions ($X_{uu} \perp N$, $X_{vv} \perp N$, so $e = g = 0$)

<u>sine-Gordon eqn</u> (taken from Rogers-Schief, Backlund transf...)

$K \equiv -1$ surfaces have coord.s s.t.

$$I = du^2 + 2\cos\theta \, du dv + dv^2 \approx \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$$

$$II = 2\sin\theta \, du dv \approx \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix}$$

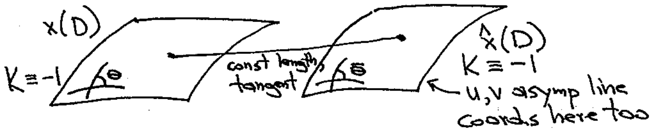
compatibility (Gauss eqn): $\theta_{uv} = \sin\theta$

↙ Kajiwara showed this

$$\left(\frac{\bar\theta-\theta}{2}\right)_u = \sin\left(\frac{\bar\theta+\theta}{2}\right)$$

$$\left(\frac{\bar\theta+\theta}{2}\right)_v = \sin\left(\frac{\bar\theta-\theta}{2}\right)$$

↖ classical Backlund transform



Exa: $\theta = 0 \longrightarrow \bar\theta = \ldots\ldots \longrightarrow \bar{\bar\theta} = \cdots$

(no surf) (pseudosphere) (Kuen surf)

This whole story can be discretized (Bob-Pink), both sine-Gordon eqn (Kajiwara, Kakei) and surf together, keeping meaning of θ.

- Inoguchi knows well about this.
- Matsuura knows too, Rikkyou Lect. Note, 2005 (Japanese)
- Bob-Suris bk, 2008 (Kakei mentioned this)

<u>sinh-Gordon eqn</u> (taken from Fujimori-Kobayashi-R notes, but classical of course)

Take $z = u+iv$ as conf'l coord.s.

$\partial_z := \frac{1}{2}(\partial_u - i\partial_v)$, $\partial_{\bar z} := \frac{1}{2}(\partial_u + i\partial_v)$, so $dz(\partial_z) = du(\frac{1}{2}\partial_u - \frac{i}{2}\partial_v) + i\,dv(\text{''})$
$= \frac{1}{2}+\frac{1}{2} = 1$, $dz(\partial_{\bar z}) = 0$, etc.

$\langle x_u, x_u\rangle = \langle x_v, x_v\rangle$, $\langle x_u, x_v\rangle = 0 \Longrightarrow$ ($\langle\cdot,\cdot\rangle$ now bilinear ext'n)

$\langle x_z, x_z\rangle = \langle x_{\bar z}, x_{\bar z}\rangle = 0$, $\langle x_z, x_{\bar z}\rangle = 2e^{2u}$, some u.

$H = \frac{1}{8e^{2u}}\langle x_{uu}+x_{vv}, N\rangle$ and

$\langle x_z, x_{zz}\rangle = \langle x_z, x_{z\bar z}\rangle = 0$, $\langle x_{\bar z}, x_{zz}\rangle = 4u_z e^{2u}$,

$\langle N_z, N\rangle = 0$, $\langle N_z, x_{\bar z}\rangle = -\langle N, x_{z\bar z}\rangle = -2He^{2u}$,

$\langle N_z, x_z\rangle = -\langle N, x_{zz}\rangle = -Q$ (Hopf diff.)

$\Longrightarrow x_{zz} = 2u_z x_z + QN$, $x_{z\bar z} = 2He^{2u}N$, $N_z = \frac{1}{2}(-2Hx_z - Qe^{-2u}x_{\bar z})$, etc

<u>Problem:</u> $K = H^2 - \frac{1}{4}Q\bar Q e^{-4u}$.

Set $F: D \to SO_3$, $F = \left(\frac{x_u}{\|x_u\|}, \frac{x_v}{\|x_v\|}, N\right) \leftarrow$ frame

$F_z = F\cdot\mathcal{A}$, $F_{\bar z} = F\cdot\mathcal{B}$

compatibility: $F_{z\bar z} = F_{\bar z z}$, i.e. $\mathcal{A}_{\bar z} - \mathcal{B}_z - [\mathcal{A},\mathcal{B}] = 0$,

iff $4u_{z\bar z} - Q\bar Q e^{-2u} + 4H^2 e^{2u} = 0$, $Q_{\bar z} = 2H_z e^{2u}$.

away from umbilics ↙

$H = \text{const} \Rightarrow Q$ holo., and can adjust u,v s.t. $H = 2Q$,

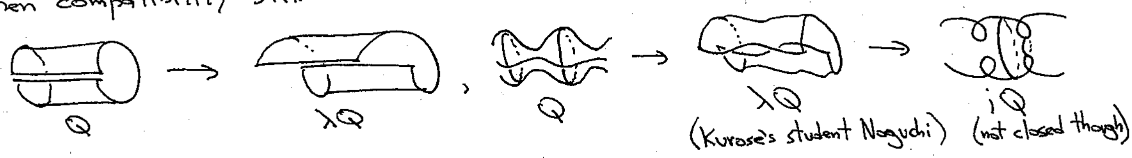
so $4u_{z\bar z} + 8H^2\cdot\frac{e^{2u}-e^{-2u}}{2} = 0$, i.e. $u_{z\bar z} = -2H^2\cdot\sinh(2u)$

essentially sinh-Gordon

$u_{z\bar z} = \frac{1}{4}(u_{uu}+u_{vv}) = \frac{1}{4}\Delta u$

same error as in poster

Can insert spectral parameter: $Q \to \lambda \cdot Q$ for $\lambda \in S^1 \subseteq \mathbb{C}$,
then compatibility still holds.



Now $F = F(z, \bar{z}, \lambda)$.

$LF = \lambda F$ ← ~~too simple, erase~~

Lax pair (keishiki) (Kajiwara, Wirokkusu) Kajiwara's way

$L\Psi = \lambda \Psi$, $\frac{d\Psi}{dt} = B\Psi$ $\longrightarrow$ $\left.\begin{array}{l} \frac{dF}{dz} = F\mathcal{A} \\ \frac{dF}{d\bar{z}} = F\mathcal{B} \end{array}\right\}$ half of Lax pair (switch Ψ and B by $\Psi \to \Psi^{-1}$ or $\Psi \to \Psi^t$)

<u>Rem</u> Kajiwara wrote

sinh-Gordon: $\frac{\partial^2 \upsilon}{\partial x \partial y} = -4 \sinh \upsilon \xleftrightarrow{\upsilon = i\theta}$ sine-Gordon: $\frac{\partial^2 \theta}{\partial x \partial y} = -4 \sin\theta$

$x, y \to z, \bar{z}$ done in Ohta's talk ↓ $\frac{\partial^2 u}{\partial z \partial \bar{z}} = -\cancel{4}\sinh u$ ↓ our way ↓ $\theta_{uv} = -4 \sin\theta$

$\mathbb{R} \ni u = i\theta \in i\mathbb{R}$ $\cancel{\ast}$

So can't get simple relation between $K \equiv -1$ and $H \equiv$ const surfaces.
$\boxed{\text{Ohta really explained this}}$ ⟺ CMC surfaces are really a different story.

II) When does a surface have isothermic coord.s?
Start with curv. line coord.s ($\mathrm{I} = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$, $\mathrm{II} = \begin{pmatrix} e & 0 \\ 0 & g \end{pmatrix}$) and stretch to $\tilde{u}(u)$, $\tilde{v}(v)$.

$\langle x_u, x_v \rangle = 0 \Rightarrow \langle x_{\tilde{u}}, x_{\tilde{v}} \rangle = \langle x_u u_{\tilde{u}} + x_v v_{\tilde{u}}, \text{''} \rangle = \cdots = 0$

$\langle x_{uv}, N \rangle = 0 \Rightarrow \langle x_{\tilde{u}\tilde{v}}, N \rangle = \langle (x_u u_{\tilde{u}} + x_v v_{\tilde{u}})_{\tilde{v}}, N \rangle = \boxtimes \cdot \langle x_{uv}, N \rangle = 0$

So $\tilde{u}, \tilde{v}$ are also curvature line coordinates. Stretch until $\tilde{E} = \tilde{G}$ if possible,
$\tilde{\mathrm{I}} = \begin{pmatrix} \tilde{E} & 0 \\ 0 & \tilde{G} \end{pmatrix}$, $\tilde{\mathrm{II}} = \begin{pmatrix} \tilde{e} & 0 \\ 0 & \tilde{g} \end{pmatrix}$.
i.e. we want $E \cdot (u_{\tilde{u}})^2 = G \cdot (v_{\tilde{v}})^2$, so $\boxed{x \text{ iso'c} \Leftrightarrow \frac{E}{G} = \frac{a(u)}{b(v)}}$

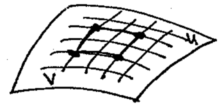
Now, for the benefit of the discrete case:
Take curv. line coord.s u, v. Set

$$q_\epsilon = (x(u+\epsilon, v) - x(u,v))(x(u+\epsilon, v+\epsilon) - x(u+\epsilon, v))^{-1} \cdot (x(u, v+\epsilon) - x(u+\epsilon, v+\epsilon))(x(u,v) - x(u, v+\epsilon))^{-1}$$

↖ not commutative!

$= cr(\sqcup, \sqcup, \sqcup, \sqcup)$

↖ using $\mathrm{Im}\,\mathbb{H}$: $x = (x_1, x_2, x_3) = x_1 i + x_2 j + x_3 k$



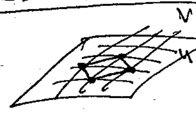
<u>Lemma</u> $\lim_{\epsilon \to 0} q_\epsilon = \frac{-E}{G}$.

Pf/ $\langle x_u, x_v \rangle = 0 \Leftrightarrow x_u x_v = -x_v x_u \Leftrightarrow x_u \cdot \frac{x_v}{x_v^2} = \frac{-x_v}{x_v^2} \cdot x_u \Leftrightarrow$
$x_u x_v^{-1} = -x_v^{-1} x_u$, so $x_u x_v^{-1} x_u x_v^{-1} = \frac{-x_u^2}{x_v^2} = -\frac{E}{G}$ //

Cor x iso'c $\iff \lim_{\epsilon\to 0} q_\epsilon = \frac{a(u)}{b(v)}$.

Rem Cor is stated without stretching u & v, useful in discrete case

Bob-Pink lemma $q_\epsilon^d = cr\big(x(u-\epsilon, v-\epsilon), x(u+\epsilon, v-\epsilon), x(u+\epsilon, v+\epsilon), x(u-\epsilon, v+\epsilon)\big)$.



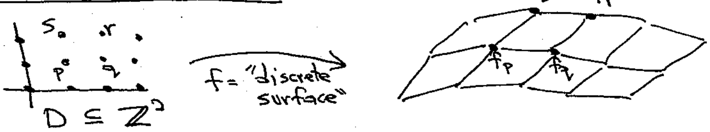
Then (u,v) conf'l $\iff q_\epsilon^d = -1 + \mathcal{O}(\epsilon)$
(u,v) iso'c $\iff q_\epsilon^d = -1 + \mathcal{O}(\epsilon^2)$

Pf/ WLOG $x(u,v) = \vec{0}$, then $x(u+\epsilon, v+\epsilon) = \epsilon x_u + \epsilon x_v + \frac{\epsilon^2}{2}(x_{uu} + x_{vv} + 2x_{uv}) + \mathcal{O}(\epsilon^3)$, etc

$$\Rightarrow q_\epsilon^d = \underbrace{x_u x_v^{-1} x_u x_v^{-1}}_{\substack{-1 \\ (\text{conf'l})}} + \epsilon(x_u x_v^{-1} x_{uv} x_v^{-1} + x_u x_v^{-1} x_u x_v^{-1} x_{uv} x_v^{-1} - x_{uv} x_v^{-1} x_u x_v^{-1} - x_u x_v^{-1} x_{uv} x_v^{-1} x_u x_v^{-1}) + \mathcal{O}(\epsilon^2)$$

so $q_\epsilon^d = -1 + \epsilon(x_u x_v x_{uv}(x_u + x_v) + x_u^2 x_{uv}(x_u - x_v)) + \mathcal{O}(\epsilon^2)$
$= -1 + \mathcal{O}(\epsilon^2)$ if iso'c coord.s, $f = 0 \Rightarrow x_{uv} \overset{\exists}{=} \alpha_1 x_u \overset{\exists}{+} \alpha_2 x_v$. //

discrete iso'c surfaces



$q_{pqrs} = cr(f_p, f_q, f_r, f_s)$

Defn f is discrete iso'c if $\exists\, a: \{\text{edges}\} \to \mathbb{R}$ s.t.
$q_{pqrs} = \frac{a_{pq}}{a_{ps}}$ and $a_{pq} = a_{rs}$, $a_{ps} = a_{qr}$ $\forall$ faces.

Rem Analogous to smooth case $\left(\lim_{\epsilon\to 0} q_\epsilon = \frac{a(u)}{b(v)}\right)$

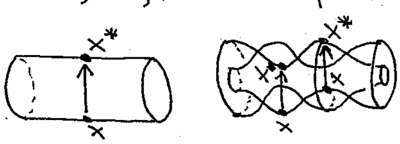
Rem B-P lemma suggests $q_{pqrs} = -1$. This was first defn, but was no good for transformation theory.

Rem $q_{pqrs} \in \mathbb{R} \Rightarrow f_p, f_q, f_r, f_s \in$ circle. This is useful: it's clear what faces are in any space form (later).

dual surfaces = Christoffel transform

Defn x^* satisfies $dx^* = x_u^{-1} du - x_v^{-1} dv$

Facts
1) x, x^* have same conf'l structure
2) x, x^* have opposite orientations
3) x, x^* have parallel tangent planes at corresponding pts



Lemma x^* exists $\iff$ x is iso'c.

Pf/ "$\Leftarrow$" assume u,v iso'c. $X_{uv} = {}^{\exists}\alpha_1 \cdot X_u + {}^{\exists}\alpha_2 \cdot X_v$.

$d(x_u^{-1}du - x_v^{-1}dv) = \frac{16}{E^2}(x_u x_{uv} x_u + x_v x_{uv} x_v)\, du \wedge dv = 0 \Rightarrow \exists x^*$.

"$\Rightarrow$" Assume x^* exists. Take curvature line coord.s for x.

Cod. eqns: $2(k_1)_v = \frac{E_v}{E}(k_2-k_1)$, $2(k_2)_u = \frac{G_u}{G}(k_1-k_2)$.

$\exists x^*$, so $\left(\frac{k_2+k_1}{k_1-k_2}\right)_u = \left(\frac{k_1+k_2}{k_2-k_1}\right)_v \Rightarrow$

$2((k_1)_{vu}+(k_2)_{uv})(k_1-k_2)^{-1} + 2(k_2-k_1)^{-2}\cdot[(k_1)_v(k_2-k_1)_u + (k_2)_u(k_2-k_1)_v] = 0$

Substitute in Codazzi eqns, get $(\log\frac{E}{G})_{uv} = 0$, i.e. $\frac{E}{G} = \frac{a(u)}{b(v)}$ //

Discrete case

We want "$df^* = f_u^{-1}du - f_v^{-1}dv$" $\leftarrow$ makes no sense,

i.e. want "$df(\partial_u)df^*(\partial_u) = 1$, $df(\partial_v)df^*(\partial_v) = -1$". Also, $\lim_{\epsilon\to 0} q_\epsilon = -1$.

In discrete case we used $q_{pqrs} = \frac{a_{pq}}{a_{ps}}$, so:

Defn f^* is Christoffel transform of f if

$$(f_q - f_p)(f_q^* - f_p^*) = a_{pq} \quad \forall \text{ edges } \overline{pq}.$$

Lemma f is discrete iso'c $\Leftrightarrow$ f^* exists.

Discrete minimal surfaces in $\mathbb{R}^3$

$g = a+ib : \mathbb{C} \xrightarrow{\text{holo.}} \mathbb{C} \Rightarrow a_u = b_v, a_v = -b_u$, so $\|(a,b)_u\| = \|(a,b)_v\|$

and $\langle (a,b)_u, (a,b)_v\rangle = 0$, so g is conformal. It's automatically parametrized by curvature lines. So it's iso'c.

So:

Defn $g_{m,n} : D \subseteq \mathbb{Z}^2 \to \mathbb{C}$ is discrete holomorphic if

$$cr(g_{m,n}, g_{m+1,n}, g_{m+1,n+1}, g_{m,n+1}) = \frac{a(m)}{b(n)}.$$

Exa's $g = c_1(m+i\cdot n)$, $g = e^{c_1 m + i c_2 n}$, $c_j \in \mathbb{R}$.

$g = (c_1(m+i\cdot n))^p \leftarrow$ no! $\rightarrow$ Agafonov's way

easy to make many examples numerically

Smooth iso'c min surf.s (W. rep)

$f = \mathrm{Re}\int (2g, 1-g^2, i+ig^2)\frac{dz}{g'}$ $\leftarrow$ iso'c

$\Leftrightarrow f_{u/v} = (i-gj)j\frac{\pm 1}{g_{u/v}}(i-gj)$

discrete min surf.s

Defn $f_q - f_p = (i-g_p j) j \frac{\alpha_{pq}}{g_q - g_p}(i - g_q j)$

$\alpha_{pq} = a(m)$ or $b(n)$

not symmetric in p and q, just made a choice here. (Kakei made a similar comment.)

Smooth iso'c CMC 1 surf.s in $\mathbb{H}^3$

$$dF = F\begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix}\frac{dz}{g'}, \quad F \in SL_2\mathbb{C}$$

$$X = F\cdot\bar{F}^t = \begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix}$$

$$\approx (x_0,x_1,x_2,x_3) \in \mathbb{H}^3 =$$

$$= \{(x_0,x_1,x_2,x_3)\in\mathbb{R}^{3,1},\ x_0^2-x_1^2-x_2^2-x_3^2=1,\ x_0>0\}$$

$$\approx \frac{(x_1,x_2,x_3)}{1+x_0} \quad \text{(Poincare ball model)}$$

discrete CMC 1 in $\mathbb{H}^3$

$$F_q - F_p = F_p\begin{pmatrix} g_p & -g_p g_q \\ 1 & -g_q \end{pmatrix}\frac{\alpha_{pq}}{g_q-g_p}$$

$f_p = F_p\cdot\bar{F}_p^t$? (no, $\det F_p \neq 1$)

Udo Jeromin defined f_p by light cone model (later)

↓ [HRSY]

Thm $f_p = \frac{1}{\det F_p}F_p\cdot\bar{F}_p^t$.

III) $\mathbb{R}^{4,1} = \{(x_1,\dots,x_5) \mid x_j\in\mathbb{R},\ \langle(x_1,\dots,x_5),(y_1,\dots,y_5)\rangle = x_1y_1+\dots+x_4y_4-x_5y_5\}$

H = quat's, $\mathrm{Im}H$ = imaginary quat.s

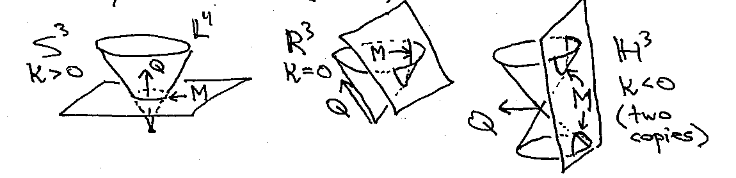
$$(x_1,\dots,x_5) \overset{1-1}{\longleftrightarrow} x_1\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}+x_2\begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}+x_3\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}+x_4\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}+x_5\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} x & x_\infty \\ x_0 & -x \end{pmatrix}$$

$x = x_1 i + x_2 j + x_3 k \in \mathrm{Im}H$, $x_0 = x_5 - x_4$, $x_\infty = x_5 + x_4$. induced metric

So $\mathbb{R}^{4,1} = \{X = \begin{pmatrix} x & x_\infty \\ x_0 & -x \end{pmatrix} \mid x\in\mathrm{Im}H,\ x_0,x_\infty\in\mathbb{R},\ \langle X,Y\rangle\cdot I = \frac{-1}{2}(XY+YX)\}$.

$\mathbb{L}^4 = \{X\in\mathbb{R}^{4,1} \mid X^2 = 0\}$ ← dim = 4.

For any $Q\in\mathbb{R}^{4,1}\setminus\{\vec{0}\}$, define $M = \{X\in\mathbb{L}^4 \mid \langle X,Q\rangle = -1\}$



WLOG $Q = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix}$.

Then $M = \{X = \frac{2}{1-\kappa x^2}\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}\} \approx \{(x_1,x_2,x_3)\in\mathbb{R}^3 \cap \{\infty\} \mid x_1^2+x_2^2+x_3^2 = -\kappa^{-1}\}$

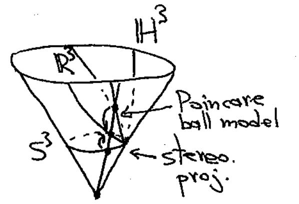
($\kappa>0 \Rightarrow$ no condition, $\kappa=0 \Rightarrow$ remove 1 pt, $\kappa<0 \Rightarrow$ remove an S^2)

$$X' = \mathcal{J}_{x'}, \text{ where } \mathcal{J}_a := \frac{2}{(1-\kappa x^2)^2}\begin{pmatrix} a+\kappa x a x & -xa-ax \\ \kappa(xa+ax) & -a-\kappa x a x \end{pmatrix}, \quad a\in\mathrm{Im}H$$

$$\langle X',\mathring{X}\rangle = \frac{4}{(1-\kappa x^2)^2}(x_1'\mathring{x}_1+x_2'\mathring{x}_2+x_3'\mathring{x}_3), \text{ so metric is } ds_\kappa^2 = \frac{4ds^2_{Eucl}}{(1-\kappa x^2)^2}$$

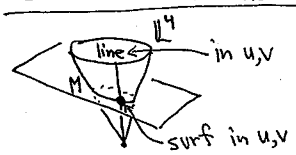
(note that $x^2<0$!)

$\Longrightarrow$ sectional curvature is κ.



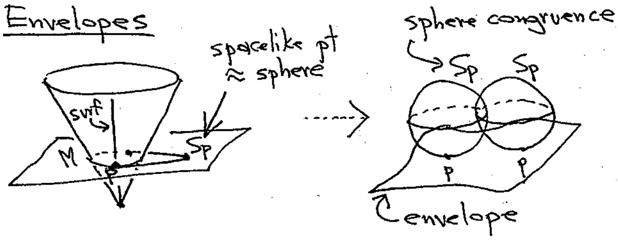
$\Longrightarrow$ $\mathbb{R}^3, \mathbb{H}^3, S^3$ Moebius equivalent

To define surf.s in space forms:



surf defined before sp form is even chosen

Envelopes



$T_x M = \{ \tilde{\mathcal{L}}_a \mid a \in \mathrm{Im}\,\mathbb{H} \}$, normal n to surface is

$n = \frac{1-Kx^2}{2} \cdot \frac{X_u \times_{\mathbb{R}^3} X_v}{\| X_u \times_{\mathbb{R}^3} X_v \|_{\mathbb{R}^3}} \approx \tilde{\mathcal{L}}_n \in T_x M$. Can assume u,v iso'c,

i.e. x is iso'c. Can compute H, and can then prove this:

<u>Defn</u> x has a <u>linear conserved quantity</u> (l.c.q.) if

$\exists\, Q_{2\times 2}, Z_{2\times 2} \in \mathbb{R}^{4,1}$ s.t.

$(\star_1)$ $\quad dP + \lambda\tau P - P\lambda\tau = 0$, $P := Q + \lambda Z$, $\forall \lambda \in \mathbb{R}$,

where $\tau = \binom{x}{1} dx^* (1, -x)$ with Christ. transf. $dx^* = x_u^{-1} du - x_v^{-1} dv$.

<u>Thm</u> x has l.c.q. iff x is CMC in some space form.

What's going on here? :

- Q gives space form
- τ is "$\underbrace{\text{logarithmic derivative of Calapso transf.}}_{\text{Lawson correspondence}}$"
- Eqn $(\star_1)$ is constancy of Calapso transf.

<u>Defn</u> Define T by $T^{-1} dT = \lambda\tau$. Then Calapso transf (T-transf, conf'l deformation) is $X \to T X T^{-1}$

($X \in \mathbb{L}^4 \Rightarrow T X T^{-1} \in \mathbb{L}^4$).

<u>Remark</u>
- Calapso transf preserves conf'l structure.
- Calapso transf is really a notion in Möbius geometry.

<u>Lemma</u> If x is iso'c, then Calapso transf exists.

Setting $P = Q + \lambda Z$, then $dP + \lambda\tau P - P\lambda\tau = 0 \iff$

$dP + T^{-1}dT P - P T^{-1} dT = 0 \iff d(TPT^{-1}) = 0 \iff$

$T(Q + \lambda Z) T^{-1}$ is constant.

This last statement is the one we will discretize, in order to define discrete CMC surfaces.

<u>discrete case</u>

- Calapso transf is $T_q = T_p(1 + \lambda \tau_{pq})$, $\tau_{pq} = \binom{f_p}{1}(f_q^* - f_p^*)(1, -f_q)$.
- <u>Lemma</u> If f is discrete iso'c, then T exists.

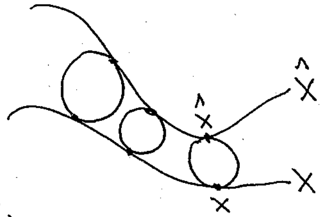
Now TPT^{-1} is constant $\iff T_q(Q_q + \lambda Z_q) T_q^{-1} = T_p (Q_p + \lambda Z_p) T_p^{-1}$

$\forall$ edges $\overline{pq}$ $\iff$

$(\star_2) \quad (1+\lambda\tau_{pq})(Q_q+\lambda Z_q) = (Q_p+\lambda Z_p)(1+\lambda\tau_{pq})$.

<u>Defn</u> f is <u>discrete CMC</u> in the space form given by Q iff $(\star_2)$ holds $\forall \lambda \in \mathbb{R}$.

<u>Darboux transformations</u>



<u>Defn</u> $\hat{x}$ is a Darboux transf. of x if

- $\exists$ sphere congruence between x and $\hat{x}$,
- the correspondence preserves curvature lines,
- the correspondence preserves conformality.

<u>Fact</u> For x and $\hat{x}$ with lifts X and $\hat{X}$, $\hat{x}$ is a Darboux transf. of x if $T\hat{X}T^{-1}$ is constant in $\mathbb{PL}^4$ for some choice of λ.

<u>Fact</u> This is equivalent to the Riccati eqn $d\hat{x} = \lambda(\hat{x}-x)dx^*(\hat{x}-x)$.

<u>Defn</u> If x is CMC with l.c.q. P, then a Darboux transf that makes $T\hat{X}T^{-1}$ constant for $\lambda=\mu$ is a Backlund transf if $\hat{X} \perp P(\mu)$.

<u>Thm</u> (Jeromin–Pedit) x is CMC in $\mathbb{R}^3$ iff some Christ. transf. of x is also a Darboux transf.

<u>Defn</u> An iso'c surface is <u>special of type n</u> if $\exists$ polynomial conserved quantity $P = Q + \lambda P_1 + \lambda^2 P_2 + \dots + \lambda^{n-1}P_{n-1} + \lambda^n Z$ s.t. $dP = -\lambda P\tau + P\lambda\tau$.

<u>Lemma</u> x of type $n \Rightarrow$ Darboux transf $\hat{x}$ is of type at most $n+1$, and then $\hat{x}$ is of type at most n iff $\hat{x}$ is a Backlund transformation.

Like the story before, all of this can be discretized, and the last theorem becomes the original definition of discrete CMC surfaces in $\mathbb{R}^3$, and we can check that the original definition is equivalent to the l.c.q. definition here.